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VIBRATIONS OF BEAMS PRESTRESSED BY INTERNAL FRICTIONLESS CABLES

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A greased cable is positioned along a generic curve in the interior of a beam and is anchored to the beam at its extreme points. The absence of friction permits a relative slipping between the cable and beam at intermediate points and this makes the cable strain dependent on the global deformation of the beam. Such a system permits controlling the beam state of stress and strain by assigning a predetermined traction force on the cable. This paper proposes a formulation of the system dynamics by defining the balance conditions describing the infinitesimal motions in the neighbourhood of a known balanced static configuration in order to evaluate the effect due to the presence of the stretched cable on the motion and free vibrations of the system. The kinematical model adopted for the beam permits a sufficiently accurate description of the behavior of thin walled beams and the cable strain is obtained as a functional of global deformation. Some qualitative aspects concerning the problem formulation and the dynamical behavior are stated. An applicative example referring to a case of interest shows that the state of stress obtained by stretching the cable notably influences only a reduced set of vibration modes, determined by the path geometry and cable force. Such aspects of the problem can be of interest in structural identification.

1. INTRODUCTION

The paper examines a system consisting of a three-dimensional rod crossed by a cable that can slip without friction in the interior of a protective tube positioned inside the rod. The cable is anchored to the rod at its extreme points and traction is usually induced in the cable thus determining an interaction between the components. Such a coupled system may find many applications in engineering because opportune geometry and positioning of the cable path permit controlling the state of the rod in order to induce stress fields opposite the stress fields due to external loads or to induce special deformation fields. On the other hand, the cable traction force is influenced by the rod deformation and the measure of its strain or stress can furnish information about rod deformation, so that the cable can be used as an actuator or as a sensor. The analysis of dynamics is of particular interest in structural engineering where it may be necessary to have active control of rod vibrations [1, 2] and where it is

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often necessary to have information regarding the effective force existing in the internal, inaccessible cable, from non-destructive tests [3, 4]. Existing literature does not furnish satisfying models, and reliable descriptions of the system regard static or quasi-static cases only [5–9].

The formulation proposed in this paper intends to provide a model for analyzing infinitesimal motions which can occur in the neighbourhood of a static balanced configuration in order to evaluate the effect of the state of stress produced by the interaction between cable and rod, eventually produced by external forces, on the dynamical response of the system. The case of infinitesimal motion around a natural (stress free) configuration, which can be obtained from the present formulation as a special case, is of interest when second order effects due to stress can be neglected.

The analysis is performed on the basis of a previous study describing the non-linear and linearized system kinematics [6, 7] for a generic three-dimensional body. In particular, cable and tube are modelled as uni-dimensional manifolds and the interaction between cable and rod is interpreted as a global constraint [10] exerted by rod deformations on cable deformations. The analysis examines the case of a cylindrical body (rod) by introducing a kinematical model which permits expressing deformation by means of longitudinal variable functions. The model assumes that the cross-sections are transversally undeformable while warping is furnished by a set of shape functions. It is necessary to consider an adequate model for warping because shear strains may play a fundamental role in such systems where the cross-sections are often thin-walled. This rod model was previously adopted by authors in a study on stability [8, 9] and it is only synthetically described here. The balance conditions are obtained by starting from the D'Alembert principle. The formulation of the problem in terms of global conditions is more natural and convenient in the problem considered because local equilibrium cannot be expressed in terms of local kinematical quantities, as usual, in that cable slipping introduces an inseparable coupling between strain and global deformation. For the rod kinematical model considered, cross-section equilibrium conditions can be equally obtained by integrating by parts, but this leads to integro-differential equations, as a consequence of the previously mentioned coupling.

The authors developed a procedure, based on classical variational methods, for modal analysis with the aim of analyzing realistic situations. A thin walled beam prestressed by a parabolic cable is finally analyzed and some aspects with interest in structural identification have been evidenced. With regard to the identification of the cable force via non-destructive tests, it can be observed that in the past scarce knowledge of the problem led to mistaken conclusions deriving from the attempt to deduce cable force from modes unaffected by its presence [3].

2. DEFORMATION ANALYSIS

The structural system is constituted by a beam, with constant cross-section and straight axis, and a cable slipping at its interior along a frictionless path. The cable is anchored at two points at the end sections of the beam.

A convenient parameterization of the problem can be achieved by considering an orthonormal reference frame $\{0; x_1, x_2, \zeta\}$ such that, in the undeformed configuration, the axis of the cross-section centroids is superimposed on the ζ axis. The generic beam material point P is therefore identified by the position vector

$$\mathbf{P}(\mathbf{x}, \zeta) = \mathbf{x} + \zeta\mathbf{A}_3 = x_\gamma\mathbf{A}_\gamma + \zeta\mathbf{A}_3, \qquad (\mathbf{x}, \zeta) \in D \times [0, L] = V \tag{1}$$

(hereinafter repeated indexes denote summation, lower-case Greek indexes assume the values 1, 2, lower-case Latin indexes assume the values 1, 2, 3) where $D \subset \mathbb{R}^2$ represents the beam cross-section and $\{\mathbf{A}_i;\, i = 1, 2, 3\}$ is an orthonormal basis. The cable is anchored at its ends and free to slip along a tube rigidly linked to the beam. It is assumed that the tube has a transversal dimension much smaller than its total length and can be modelled as a curve coinciding with its physical axis. In the reference configuration the tube position is described by a smooth curve parameterized in ζ, traced by

$$\mathbf{H}(\zeta) = \mathbf{x}_c(\zeta) + \zeta\mathbf{A}_3 = x_{c_\gamma}(\zeta)\mathbf{A}_\gamma + \zeta\mathbf{A}_3, \qquad \zeta \in [0, L]; \tag{2}$$

where $\mathbf{x}_c$ is a known function of ζ which identifies the crossing point of the cable in the generic beam cross-section. By denoting with *primes* the total derivative with respect to ζ, the tangent unit-vector to such a path is given by

$$\mathbf{G} = \frac{\mathbf{H}'}{|\mathbf{H}'|} = \frac{x'_{c_\gamma}\mathbf{A}_\gamma + \mathbf{A}_3}{\sqrt{x'_{c_\gamma}x'_{c_\gamma} + 1}}, \tag{3}$$

and its length is expressed by the quantity

$$\Lambda = \int_0^L |\mathbf{H}'|\, \mathrm{d}\zeta = \int_0^L \sqrt{x'_{c_\gamma}x'_{c_\gamma} + 1}\, \mathrm{d}\zeta. \tag{4}$$

The deformation and strain of system components will now be analyzed. Under the hypothesis that the beam cross-section is rigid in its own plane, the admissible displacements of the beam can be represented by [8, 11]

$$\mathbf{u}(\mathbf{x}, \zeta;\, t) = \mathbf{u}_0(\zeta;\, t) + \boldsymbol{\varphi}(\zeta;\, t) \times \mathbf{x} - \mathbf{I}_3\boldsymbol{\varphi}(\zeta;\, t) \times \mathbf{x}_s + [\boldsymbol{\chi}(\zeta;\, t) \cdot \boldsymbol{\psi}(\mathbf{x})]\mathbf{A}_3 \qquad (\mathbf{x}, \zeta;\, t) \in V \times [t_0, \infty), \tag{5}$$

where $\mathbf{x}_s$ is the vector identifying the cross-section shear centre, $\mathbf{I}_3 = \mathbf{A}_3 \otimes \mathbf{A}_3$ is the projector along $\mathbf{A}_3$, $\mathbf{u}_0 = u_{0_i}\mathbf{A}_i$ and $\boldsymbol{\varphi} = \varphi_i\mathbf{A}_i$ respectively describe rigid translations and rotations of the cross-section, $(\boldsymbol{\chi} \cdot \boldsymbol{\psi})\mathbf{A}_3$ describes the warping displacements. In particular, the last term is obtained as a scalar product between the vectors $\boldsymbol{\chi}$ which group the intensities of a generic number of warping functions contained in $\boldsymbol{\psi}$. The warping functions fulfil orthogonality conditions [8, 12] and four different warping functions (pure and non-uniform torsion warping and shear warpings) have been considered in applications. More simple kinematical models, as Kirchhoff, Timoshenko or Vlasov models [13–15], can be obtained as special cases.

In the following, for the sake of simplicity, the three unknown vectors $\mathbf{u}$, $\boldsymbol{\varphi}$ and $\boldsymbol{\chi}$ will be grouped in the unique vector $\mathbf{v} = [\mathbf{u}_0, \boldsymbol{\varphi}, \boldsymbol{\chi}]$ and $C = \{\mathbf{v} : [0, L] \rightarrow \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^{1+n}\}$ is the functional space of the admissible beam deformations.

According to equation (5), the acceleration of the generic beam point results

$$\ddot{\mathbf{u}}(\mathbf{x}, \zeta; t) = \ddot{\mathbf{u}}_0(\zeta; t) + \ddot{\boldsymbol{\varphi}}(\zeta; t) \times \mathbf{x} - \mathbf{I}_3 \ddot{\boldsymbol{\varphi}}(\zeta; t) \times \mathbf{x}_s + [\ddot{\boldsymbol{\chi}}(\zeta; t) \cdot \boldsymbol{\psi}(\mathbf{x})]\mathbf{A}_3. \tag{6}$$

Deformation is measured by the displacement field gradient $\mathbf{U}$

$$\begin{aligned}\mathbf{U}(\mathbf{x}, \zeta;\ t) = [&\mathbf{u}_0' - \boldsymbol{\varphi} \times \mathbf{A}_3 + \boldsymbol{\varphi}' \times \mathbf{x} - \mathbf{I}_3 \boldsymbol{\varphi}' \times \mathbf{x}_s + (\boldsymbol{\chi}' \cdot \boldsymbol{\psi})\mathbf{A}_3] \\ &\otimes \mathbf{A}_3 + \mathbf{A}_3 \otimes \nabla_D(\boldsymbol{\chi} \cdot \boldsymbol{\psi}) + (\boldsymbol{\varphi} \times \mathbf{A}_i) \otimes \mathbf{A}_i\end{aligned} \tag{7}$$

and by the linear strain tensor $\mathbf{E} = (\mathbf{U} + \mathbf{U}^T)/2$. The expression $\nabla_D(\cdot) = (\cdot)_{,\gamma}\mathbf{A}_\gamma$ denotes the gradient of scalar functions defined on the cross-section D.

The motion of the tube rigidly linked to the beam, according to equation (5), can be obtained as

$$\mathbf{h}(\zeta;\ t) = \mathbf{H} + \mathbf{u}_0 + \boldsymbol{\varphi} \times \mathbf{x}_c - \mathbf{I}_3 \boldsymbol{\varphi} \times \mathbf{x}_s + [\boldsymbol{\chi} \cdot \boldsymbol{\psi}^H(\zeta)]\mathbf{A}_3; \tag{8}$$

hereinafter, apex H indicates the composition of the generic function with the vector describing the configuration of the cable (e.g., $\boldsymbol{\psi}^H(\zeta) = \boldsymbol{\psi}(\mathbf{x}) \circ \mathbf{H}(\zeta)$). Consequently, the tangent vector $\mathbf{H}'$ transforms into

$$\begin{aligned}\mathbf{h}'(\zeta;\ t) = &\mathbf{H}' + \mathbf{u}_0' + \boldsymbol{\varphi}' \times \mathbf{x}_c + \boldsymbol{\varphi} \times \mathbf{x}_c' - \mathbf{I}_3 \boldsymbol{\varphi}' \times \mathbf{x}_s \\ &+ [(\boldsymbol{\chi}' \cdot \boldsymbol{\psi}^H) + \nabla_D(\boldsymbol{\chi} \cdot \boldsymbol{\psi}^H) \cdot \mathbf{H}']\mathbf{A}_3,\end{aligned} \tag{9}$$

and the length of the deformed tube results

$$\begin{aligned}\lambda(t) = \int_0^L &\mathbf{H}' + \mathbf{u}_0' + \boldsymbol{\varphi}' \times \mathbf{x}_c + \boldsymbol{\varphi} \times \mathbf{x}_c' - \mathbf{I}_3 \boldsymbol{\varphi}' \times \mathbf{x}_s \\ &+ [(\boldsymbol{\chi}' \cdot \boldsymbol{\psi}^H) + \nabla_D(\boldsymbol{\chi} \cdot \boldsymbol{\psi}^H) \cdot \mathbf{H}']\mathbf{A}_3 \,\mathrm{d}\zeta.\end{aligned} \tag{10}$$

So far, the kinematics of the beam and that of the tube have been described. Obviously, since the latter is a material curve of the beam, its kinematics are fully described by the same functions describing beam motion. With regard to the cable which is a geometric entity distinct from the beam, it is instead necessary to introduce a suitable deformation function to describe, at each time, the position of its material points. The coupling with the beam induces a global constraint on the cable motion which is determined once the beam kinematics descriptors and a further scalar function have been fixed. In this case, the determination of the places occupied by the cable material points is however a secondary problem which does not have a simple solution and which is of little interest in applications, while its major interest is in evaluating the local cable strain which can be measured by means of the ratio a between the final and initial length of an infinitesimal portion of cable. From the assumption of frictionless contact and of negligible mass for the cable it follows that the cable stress is constant as a consequence of the equilibrium along the path tangent. The further hypothesis of homogeneous cable, ensures that the strain a is uniform along the cable and can easily be calculated as the ratio between the total tube length in the actual configuration $\lambda(t)$ and its initial total length Λ

$$a(t) = \frac{\lambda}{\Lambda} = \frac{1}{\Lambda}\int_0^L |\mathbf{h}'| \, \mathrm{d}\zeta. \tag{11}$$

Equation (11) evidences a functional dependence of a on the trace of beam deformation along the tube curve and reflects the peculiarity of the global coupling between the cable and the beam. It must be noted that a is non-zero only for beam motions inducing a variation in the length of the cable path. Coherently with the infinitesimal formulation, equation (11) can be linearized with respect to displacements, furnishing the expression

$$a \cong 1 + \frac{1}{\Lambda}\int_0^L \mathbf{G} \cdot \mathbf{U}^H \mathbf{H}' \, \mathrm{d}\zeta, \tag{12}$$

from which it follows that cable stretching is induced by the beam strain components along the tangent to the path while the skew-symmetric part of $\mathbf{U}$, describing infinitesimal rotations, provides no effects. For further details on the kinematics of the system constituted by a beam and a cable the reader may refer to references [6–8].

3. MOTION OF THE SYSTEM

Dynamics of the cable–beam system under examination can be studied as a particular case of the dynamics of a system constituted by a generic three-dimensional body and a cable with negligible mass. In the general case of finite deformations, the balance condition can be obtained in weak form by starting from the Lagrange-D'Alembert principle [16] written in the following form:

$$\int_V \mathbf{S}(\nabla \mathbf{p}) \cdot \nabla \hat{\mathbf{p}} \, \mathrm{d}V - \int_V \rho_0 (\mathbf{b} - \ddot{\mathbf{p}}) \cdot \hat{\mathbf{p}} \, \mathrm{d}V - \int_{\partial V} \mathbf{f} \cdot \hat{\mathbf{p}} \, \mathrm{d}\partial V + \tau(a) \hat{a} \Lambda = 0 \qquad \forall \hat{\mathbf{p}} \in C; \ \forall t \in (t_0, \ +\infty), \tag{13}$$

where $\mathbf{S}$ is the first Piola–Kirchhoff stress tensor relative to the body, ρ_0 is the elementary mass of the body in its reference configuration, $\mathbf{b}$ and $\mathbf{f}$ are. The mass and contact forces applied to the body and τ is the modulus of the cable force. With *hats* the admissible variations of the deformation functions are denoted. Such a global approach is the most natural and simple for the problem because, as already mentioned, the cable local strain has a functional dependence on the deformation functions describing the beam configuration. The form given to the principle does not contain coupling terms between cable and body and this expresses the absence of friction, and the presence of interaction forces which must be orthogonal to the path along which cable material points can move. As usual, the motion can be identified once the initial state of the system, furnished by the fields $\mathbf{p}$ and $\dot{\mathbf{p}}$, are known at the initial instant t_0.

The aim of this paper is to study the infinitesimal motion of the system in the neighbourhood of a static equilibrated configuration; in this case the condition describing the motion can be deduced by linearizing equation (13) with respect to the displacements measured from the known configuration. By choosing this configuration coincident with the reference body configuration, where the stress field on the beam, measured by the Cauchy stress tensor $\mathbf{T}_0$, is balanced by the interaction forces due to cable force τ_0 and by the external force fields $\mathbf{b}_0$ and $\mathbf{f}_0$, equation (13) reduces to

$$\int_V \mathbf{C}[\mathbf{E}]\cdot\hat{\mathbf{E}}\,\mathrm{d}V+\frac{c}{\Lambda}\left(\int_0^L \mathbf{G}\cdot\mathbf{U}^H\mathbf{H}'\,\mathrm{d}\zeta\right)\left(\int_0^L \mathbf{G}\cdot\hat{\mathbf{U}}^H\mathbf{H}'\,\mathrm{d}\zeta\right)+\int_V \mathbf{T}_0\cdot\mathbf{U}^T\hat{\mathbf{U}}\,\mathrm{d}V$$
$$+\tau_0\int_0^L\frac{\mathbf{I}-\mathbf{G}\otimes\mathbf{G}}{|\mathbf{H}'|}\cdot(\mathbf{U}^H\mathbf{H}'\otimes\hat{\mathbf{U}}^H\mathbf{H}')\,\mathrm{d}\zeta=\int_V\rho_0(\tilde{\mathbf{b}}-\ddot{\mathbf{u}})\cdot\hat{\mathbf{u}}\,\mathrm{d}V+\int_{\partial V}\tilde{\mathbf{f}}\cdot\hat{\mathbf{u}}\,\mathrm{d}\partial V,$$
$$\forall\hat{\mathbf{u}}\in C,\quad(14)$$

where $\mathbf{C}$ denotes the elastic tensor of the material constituting the body at the stress $\mathbf{T}_0$ and c denotes the cable stiffness at the traction force τ_0. In equation (14) the two elastic terms and the two geometrical terms relative to body and cable are balanced from the inertial term related to time derivatives of displacements and the contribution of variation of external forces denoted by $\tilde{\mathbf{b}}$ and $\tilde{\mathbf{f}}$. With regard to the elastic terms it is remarkable that, while the term relative to the body is classically obtained by integrating a quadratic form, the term relative to the cable is furnished by the product between two linear integrals as a result of the particular coupling between cable and body. It can be observed that equation (14) becomes homogeneous when the external forces do not vary from the reference fields $\mathbf{b}_0$ and $\mathbf{f}_0$ (free vibrations). The last reference external force fields however indirectly influence the system vibrations throughout the geometrical term containing $\mathbf{T}_0$ and τ_0.

The balance condition for the cable–beam system can be obtained by introducing the deformation fields of the previous section into the Lagrange–D'Alembert principle. It is important to note that the relations of the previous section provide a reduced set of deformations only, as a consequence of the internal local constraint deriving from the transversal rigidity of the cross-section, from which $E_{\alpha\beta}=0$, and as a consequence of the global constraint deriving from considering a finite number of shape functions ψ_i to describe warping. This provides a splitting of the stress tensor measuring the stress increment due to the motion in two parts: an active part which makes virtual work for the admissible strain variations and which is related to the actual deformation of the body by the constitutive relations, and a reactive part which does not make virtual work and which is necessary to impose the global constraints [10]. It is evident that such a splitting concerns only the stress produced by strain contained in the beam constitutive terms, while the entire residual stress field $\mathbf{T}_0$ must be considered in the geometrical term.

Since the displacement field is expressed as a summation of products between unknown functions defined on the beam axis abscissa ζ and known functions of the cross section co-ordinates x_α (e.g., warping functions), the volume integrals on the body can be partially developed by integrating known functions on cross-sections and obtaining dynamic quantities depending on ζ which are dual to the kinematical descriptors related to $\mathbf{v}$, as usual in beam theories. The duality relations due to the cable involve only scalar quantities which however have a functional dependence on the kinematical descriptors of the beam deformation. In such a way, the new balance condition is constituted by a summation of line integrals only. In the following, each term forming the balance condition (14) will separately be analysed.

Under the assumption that the elasticity tensor $\mathbf{C}$ furnishes the following, transversally isotropic, constitutive law

$$\mathbf{C}[\mathbf{E}] = 2G\mathbf{E} + (E - 2G)(\mathbf{E}\cdot\mathbf{I}_3)\mathbf{I}_3, \tag{15}$$

and taking into account equation (7), the beam elastic term transforms into the following, usually positive definite, bilinear form:

$$\int_V \mathbf{C}[\mathbf{E}]\cdot\hat{\mathbf{E}}\,\mathrm{d}V = \int_0^L \mathbf{K}\mathcal{D}\mathbf{v}\cdot\mathcal{D}\hat{\mathbf{v}}\,\mathrm{d}\zeta, \tag{16}$$

where $\mathbf{v}\rightarrow\mathcal{D}\mathbf{v}$ is the formal linear differential operator as $\mathcal{D}\mathbf{v} = [\mathbf{u}_0' - \boldsymbol{\varphi}\times\mathbf{A}_3, \boldsymbol{\varphi}', \boldsymbol{\chi}, \boldsymbol{\chi}']$ and $\mathbf{K}$ represents the cross-section stiffness matrix which furnishes the dynamical entity $\mathbf{K}\mathcal{D}\mathbf{v}$, dual of $\mathcal{D}\hat{\mathbf{v}}$, grouping the classical internal resultants of active stress: axial and shear forces, bending and twisting moments, bi-shears and bi-moments relative to warping displacements [8, 12].

As already mentioned, the cable elastic term, is given by the product of two functionals of body deformation as a result of the particular coupling between the two structural components. Quantity

$$\mathbf{U}^H\mathbf{H}' = \mathbf{h}' = \mathbf{u}_0' + (\varphi_\gamma\mathbf{A}_\gamma\times\mathbf{x}_c)' + [\varphi_3\mathbf{A}_3\times(\mathbf{x}_c - \mathbf{x}_s)]' + (\boldsymbol{\chi}\cdot\boldsymbol{\psi}^H)'\mathbf{A}_3, \tag{17}$$

which appears in the term, represents the difference between the elements which are tangent to the path in the deformed configuration and in the undeformed configuration but, since $\mathbf{G}$ and $\mathbf{H}'$ are parallel, only the symmetric part of the displacement gradient $\mathbf{U}^H$ (pure deformation) is significant; in other words only the component of $\mathbf{U}^T\mathbf{H}'$, which is point by point parallel to the cable path, may induce elastic effects on the cable. It must be observed that such a term is only positive semi-defined as a consequence of its analytical structure. It is, in fact, non-zero only when the beam motion induces a global stretch of the cable path. For the specific case it assumes the form

$$\frac{c}{\Lambda}\left(\int_0^L \mathbf{G}\cdot\mathbf{U}^H\mathbf{H}'\,\mathrm{d}\zeta\right)\left(\int_0^L \mathbf{G}\cdot\hat{\mathbf{U}}^H\mathbf{H}'\,\mathrm{d}\zeta\right) = \frac{c}{\Lambda}\left(\int_0^L \boldsymbol{\Theta}\cdot\mathcal{D}^H\mathbf{v}\,\mathrm{d}\zeta\right)\left(\int_0^L \boldsymbol{\Theta}\cdot\mathcal{D}^H\hat{\mathbf{v}}\,\mathrm{d}\zeta\right), \tag{18}$$

where the vector $\boldsymbol{\Theta} = [\boldsymbol{\Theta}_N, \boldsymbol{\Theta}_M, \boldsymbol{\Theta}_\upsilon, \boldsymbol{\Theta}_\mu]$ collects the following kinematical entities related to the cable path:

$$\begin{aligned} &\boldsymbol{\Theta}_N(\zeta) = \frac{\mathbf{H}'\otimes\mathbf{H}'}{|\mathbf{H}'|}\mathbf{A}_3, \quad \boldsymbol{\Theta}_M(\zeta) = \mathbf{x}_c\times\boldsymbol{\Theta}_N - \mathbf{I}_3(\mathbf{x}_s\times\boldsymbol{\Theta}_N),\\ &\Theta_{\upsilon I}(\zeta) = \boldsymbol{\Theta}_N\cdot\nabla_D\psi_I^H, \quad \boldsymbol{\Theta}_\mu(\zeta) = \boldsymbol{\psi}^H(\boldsymbol{\Theta}_N\cdot\mathbf{A}_3). \end{aligned} \tag{19}$$

The third term of equation (14) is the classical geometrical term which represents the virtual work performed by the known stress state $\mathbf{T}_0$ on the quadratic strain induced by the admissible variation of the configuration. It cannot be defined in sign and thus can penalize the previous terms depending on the elastic stiffness of the system. In such a case the global stiffness of the system may reduce and unstable motions can occur as a limit case. By making the deformation field of the cylindrical body explicit, the following relation is obtained

$$\int_V \mathbf{T}_0 \cdot \mathbf{U}^T \hat{\mathbf{U}} \, \mathrm{d}V = \int_0^L \mathbf{W} \mathcal{G}\mathbf{v} \cdot \mathcal{G}\hat{\mathbf{v}} \, \mathrm{d}\boldsymbol{\zeta}, \tag{20}$$

where the coefficient matrix $\mathbf{W}$ is related to the stress tensor components and to the geometry of the cross-section. while the linear differential formal operator $\mathbf{v} \rightarrow \mathcal{G}\mathbf{v}$ is $\mathcal{G}\mathbf{v} = [\mathbf{u}_0, \mathbf{u}_0', \boldsymbol{\varphi}, \boldsymbol{\varphi}', \boldsymbol{\chi}, \boldsymbol{\chi}']$.

The fourth term is relative to the cable and, as for the previous terms, depends on the quantity $\mathbf{U}^H \mathbf{H}'$, but this time only the component which is orthogonal to the cable path becomes significative. Since the cable is stretched and designed to maintain positive stress even under the action of the external loads, and the integrated functions are non-negative for $\hat{\mathbf{v}} = \mathbf{v}$, it derives that this term is positive semi-defined. By substituting the body deformation quantities according to the deformation model adopted, the following equation is obtained

$$\tau_0 \int_0^L \frac{\mathbf{I} - \mathbf{G} \otimes \mathbf{G}}{|\mathbf{H}'|} \cdot (\mathbf{U}^H \mathbf{H}' \otimes \hat{\mathbf{U}}^H \mathbf{H}') \, \mathrm{d}\zeta = \tau_0 \int_0^L \mathbf{W}^c \mathcal{G}^H \mathbf{v} \cdot \mathcal{G}^H \hat{\mathbf{v}} \, \mathrm{d}\zeta, \tag{21}$$

where $\mathbf{W}^c$ is a coefficient matrix depending on the geometrical path of the cable and $\mathcal{G}^H$ is the trace of the previously defined operator along the path.

The term on the right in equation (14) is related to the external forces and the mass distribution in the solid. By substituting the acceleration field according to equation (6) and the deformation field, it derives that

$$\int_V \rho_0 (\tilde{\mathbf{b}} - \ddot{\mathbf{u}}) \cdot \hat{\mathbf{u}} \, \mathrm{d}V + \int_{\partial V} \tilde{\mathbf{f}} \cdot \hat{\mathbf{u}} \, \mathrm{d}\partial V = \int_0^L (\mathbf{q} - \mathbf{L}\ddot{\mathbf{v}}) \cdot \hat{\mathbf{v}} \, \mathrm{d}\zeta, \tag{22}$$

where

$$\mathbf{L}(\zeta) = \begin{bmatrix} \mathbf{L}_{u_0 u_0} & \mathbf{L}_{u_0 \varphi} & \mathbf{0} \\ \mathbf{L}_{u_0 \varphi}^T & \mathbf{L}_{\varphi\varphi} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{L}_{\chi\chi} \end{bmatrix}, \quad \ddot{\mathbf{v}}(\zeta) = \begin{bmatrix} \ddot{\mathbf{u}}_0 \\ \ddot{\boldsymbol{\varphi}} \\ \ddot{\boldsymbol{\chi}} \end{bmatrix}. \tag{23}$$

The symmetric matrix $\mathbf{L}$ collects the mass *resultants* which are significant for the model. In particular the diagonal terms represent the masses that produce generalised inertia forces due to the linear, angular and warping accelerations; the others are coupling terms. Under the orthogonality assumption for the warping functions, the only possible coupling is the roto-translation coupling as a result of the non-coincidence between shear centre and centroid of the cross-section. Obviously, such terms are zero for double symmetric cross-sections. The vector valued function $\mathbf{q}(\zeta)$ is obtained by integrating terms related to mass force

$\rho_0\tilde{\mathbf{b}}$ on the cross-section and by integrating terms related to contact forces $\tilde{\mathbf{f}}$ on the cross-section boundary. For the sake of simplicity, it is assumed that no contact forces are applied at the basis of the cylinder.

In conclusion, the introduced operators $\mathcal{G}$, $\mathcal{D}$, $\mathbf{\Theta}$, $\mathbf{K}$, $\mathbf{W}$, $\mathbf{L}$ make it possible to formulate the problem for the kinematical model considered and the infinitesimal motion is defined once $\mathbf{v}$ and $\dot{\mathbf{v}}$ are assigned at the initial instant.

When second order effects due to stress can be neglected, the formulation of the problem can be obtained by considering only the elastic terms and starting from the natural configuration.

An alternative formulation expressing equilibrium of cross-sections can be deduced by integrating by parts, following a way similar to that traced in reference [8]. Differently from the classical beam problems, this does not furnish a really local formulation but leads to integro-differential equations that are of less interest.

4. MODAL ANALYSIS

Modal analysis describing free vibrations of the system can be performed by considering the previous variational formulation and by determining solutions in the case of absence of external forces. The system is conservative and it is possible to seek solutions in the following form

$$\mathbf{v}(\zeta;\ t) = \mathrm{e}^{\mathrm{i}\theta t}\boldsymbol{\phi}(\zeta), \tag{24}$$

where $\mathrm{i} = \sqrt{-1}$ and θ is proportional to the frequency of the vibration mode $\boldsymbol{\phi}(\zeta)$. By substituting this last expression into the balance condition and cutting off time, the following condition, involving spatial functions only, is obtained

$$\begin{aligned}\psi(\boldsymbol{\phi}, \hat{\mathbf{v}}) := \int_0^L \mathbf{K}\mathcal{D}\boldsymbol{\phi}\cdot\mathcal{D}\hat{\mathbf{v}}\,\mathrm{d}\zeta + \frac{c}{\Lambda}\left(\int_0^L \mathbf{\Theta}\cdot\mathcal{D}^H\boldsymbol{\phi}\,\mathrm{d}\zeta\right)\left(\int_0^L \mathbf{\Theta}\cdot\mathcal{D}^H\hat{\mathbf{v}}\,\mathrm{d}\zeta\right)\\ + \tau_0\int_0^L \mathbf{W}^c\mathcal{G}^H\boldsymbol{\phi}\cdot\mathcal{G}^H\hat{\mathbf{v}}\,\mathrm{d}\zeta + \int_0^L \mathbf{W}\mathcal{G}\boldsymbol{\phi}\cdot\mathcal{G}\hat{\mathbf{v}}\,\mathrm{d}\zeta - \theta^2\int_0^L \mathbf{L}\boldsymbol{\phi}\cdot\hat{\mathbf{v}}\,\mathrm{d}\zeta = 0 \quad \forall\hat{\mathbf{v}}\in C.\end{aligned} \tag{25}$$

This is an eigenvalue problem and real non-zero eigenvalues exist if the condition $\psi(\boldsymbol{\phi}, \boldsymbol{\phi}) < 0$ is fulfilled for every $\boldsymbol{\phi} \neq \mathbf{0}$ [17].

An approximate solution of the problem of cable–beam system free vibrations, can be obtained from the weak formulation previously described thanks to the classical Ritz approximation method. By considering suitable truncations of a complete sequence in the solution space, the approximated solution can be found in a finite-dimensional subspace of C.

Each jth component of vector $\mathbf{v}$ which contains the unknown functions describing system kinematics, is approximated by means of the first m_j terms of a complete sequence, in the following way

$$v_j = y_{(j)k}(\zeta)w_{(j)k} \quad (k = 1, \ldots, m_j, \text{ no summation on } j). \tag{26}$$

By properly grouping functions $y_{(j)k}$ and the unknown coefficients $w_{(j)k}$ in the matrix of shape functions $\mathbf{Y}(\zeta)$ and in the vector $\mathbf{w}$, respectively, equation (26) can be rewritten in the form $\mathbf{v} = \mathbf{Y}\mathbf{w}$. In the case of homogeneous boundary conditions, the problem is linear and the admissible variations of the displacements can be written in the form $\hat{\mathbf{v}} = \mathbf{Y}\hat{\mathbf{w}}$. By substituting, the eigenvalue problem reduces to

$$(\mathbf{A} - \vartheta^2\mathbf{M})\mathbf{w}\cdot\hat{\mathbf{w}} = 0, \quad \forall\hat{\mathbf{w}} \neq \mathbf{0}, \tag{27}$$

where matrixes $\mathbf{A}$ and $\mathbf{M}$ have the expressions

$$\mathbf{A} = \int_0^L (\mathcal{D}^*\mathbf{Y})^T\mathbf{K}(\mathcal{D}^*\mathbf{Y})\,\mathrm{d}\zeta + \frac{c}{\Lambda}\left(\int_0^L (\mathcal{D}^*\mathbf{Y})^T\mathbf{\Theta}\,\mathrm{d}\zeta\right)\left(\int_0^L (\mathcal{D}^*\mathbf{Y})^T\mathbf{\Theta}\,\mathrm{d}\zeta\right)^T + \int_0^L (\mathcal{G}^*\mathbf{Y})^T(\tau_0\mathbf{W}^c + \mathbf{W})(\mathcal{G}^*\mathbf{Y})\,\mathrm{d}\zeta, \tag{28a}$$

$$\mathbf{M} = \int_0^L \mathbf{Y}^T\mathbf{L}\mathbf{Y}\,\mathrm{d}\zeta. \tag{28b}$$

Operators $\mathcal{D}^*$ and $\mathcal{G}^*$ furnish matrixes by mapping every column $\mathbf{y}_k^* = [Y_{1k}, \ldots, Y_{10k}]$ of $\mathbf{Y}$ into $\mathcal{D}\mathbf{y}_k^*$ and $\mathcal{G}\mathbf{y}_k^*$. Equation (27) implies that the

$$(\mathbf{A} - \vartheta^2\mathbf{M})\mathbf{w} = \mathbf{0}. \tag{29}$$

must be fulfilled. The solution of the eigenvalue problem makes it possible to obtain the approximated values of the most meaningful eigenperiods and vibration modes.

5. VIBRATIONS OF THIN WALLED BEAMS WITH PARABOLIC CABLE

5.1. ANALYZED PROBLEM

The model previously introduced is employed to study the free vibrations of a class of concrete beams prestressed by means of slipping internal cables in order to test the influence of the elastic characteristics of the materials, as well as the influence of the path and the prestressing cable stress, on the beam free vibrations. Numerical applications have been developed in order to show some characteristic aspects of the modal properties of these systems and to analyze only one of the typologies usually adopted in structural engineering. The same qualitative aspects described in the following pages have been however observed in other usual situations not reported here, such as beams with T-section or I-section with wide flange.

The numerical applications refer to a simply supported beam with the end twisting rotations prevented by suitable restraints which permit cross-section warping. The depth $H = 1300$ mm of the cross-section and the elastic modulus $E_0 = 30\,000$ N/mm^2 of the beam have been assumed as fixed quantities while the other geometrical and constitutive parameters may vary. In particular the dimensions of the considered "I" cross-section have been parameterized as described in Figure 1. The cable stiffness is $c = 0{\cdot}1E_0A$ where A is the area of the beam cross-section.

Prestressing is carried out by means of a parabolic cable, lying on the vertical symmetry plane of the cross-section, whose path is described by

$$\mathbf{H}(\zeta) = \left\{(2H - 4s)\left[\left(\frac{\zeta}{L}\right) - \left(\frac{\zeta}{L}\right)^2\right] + s\right\}\mathbf{A}_2 + \zeta\mathbf{A}_3, \tag{30}$$

where s is the cable eccentricity at the ends of the beam where anchorages are located. Such a geometry produces interaction transversal forces opposite to external loads and is usually adopted for reducing the extreme values of bending and shear stresses in the beam.

The need to have meaningful results clearly explainable justifies the simplicity of the example geometry (symmetry of the cross-section and of the cable path).

The problem has been analyzed by considering the kinematical model proposed in reference [13] (*LS model*), which makes it possible to accurately describe the kinematics of thin-walled beams. The kinematical descriptors consist of ten functions of the beam axis abscissa: six describing the rigid motion of the cross-section and the others describing intensities of the warping displacements due to primary and non-uniform torsion, as well as to shear forces (shear lag effect). In Figure 2 the qualitative description of the shape of the warping functions used for the considered "I" section are reported: ψ_0 and ψ_3 concern torsion while ψ_1 and ψ_2 concern shear in the x_1 and x_2 directions.

The importance of shear strain on the vibration of non-prestressed thin walled beams is already known; numerical results and comparisons with a simplified model are reported in reference [13]. It is reasonable to assume that the shear strain maintains a notable influence even in the studied case of prestressed beam, so that such a refined model has been adopted in the analysis. However, a comparison with the results derived from the simplified Kirchhoff–Vlasov model (*KV model*) will be carried out in the following in order to exploit the necessity of the previous, more complex, model. As is well known, the *KV model* neglects the angular strain due to shear and the warping related to the non-uniform torsion stress.

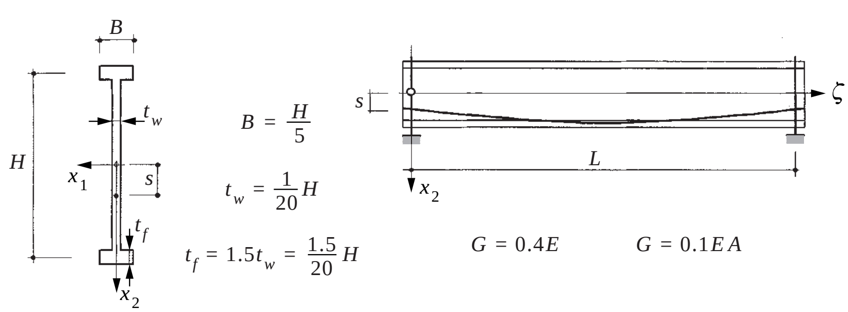

Figure 1. Geometry of the problem and profile of the cable path.

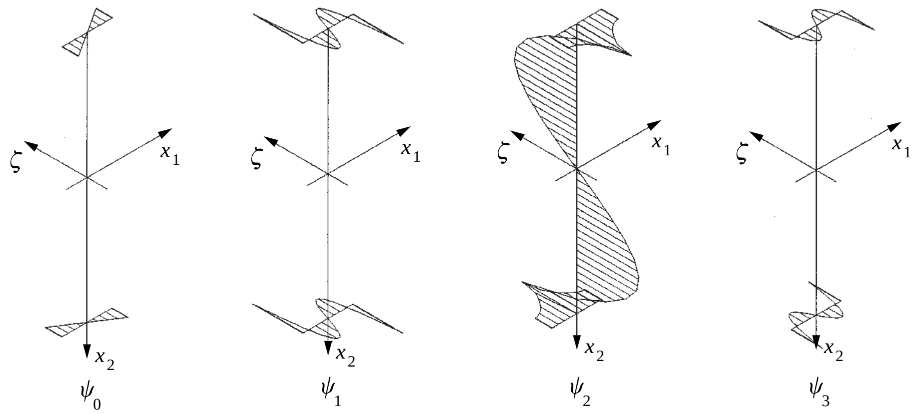

Figure 2. Warping shape functions for the cross-section adopted.

The variational format of the problem and the regularity of stiffness and inertial properties, together with the regularity of the cable path make the Ritz approximation method efficient in seeking a numerical solution. Truncations of the sine normalised sequence are assumed as shape functions for the beam axis displacements (u_1, u_2) and the twisting rotations (φ_3) while for the shape functions describing the bending rotations (φ_1, φ_2) as well as the intensities of the warping (χ_0, χ_1, χ_2, χ_3), truncations of the Tschebichef polynomial sequence are assumed. With regard to the axial translation of the cross-section (u_3), a linear function is joined to a sinusoidal expansion in order to fulfil the boundary conditions. Table 1 reports the expressions of the adopted shape functions.

TABLE 1

Ritz function sequences adopted

$u_1 u_2 \varphi_3$	$f_i(\zeta) = \sqrt{\frac{2}{L}} \sin\left(\frac{i\pi\xi}{L}\right) \quad i = 1, 2, \ldots$
u_3	$f_1(\zeta) = \frac{\zeta}{L}\sqrt{\frac{3}{L}}$ $f_i(\zeta) = \sqrt{\frac{2}{L}} \sin\left[\frac{(i-1)\pi\zeta}{L}\right] \quad i = 2, \ldots$
$\varphi_1 \varphi_2$ $\chi_0\ \chi_1\ \chi_2\ \chi_3$	$f_1 = 1$ $f_2 = \zeta$ $f_i = 2f_{i-1}\zeta - f_{i-2} \quad i = 3, 4, \ldots$

5.2. INFLUENCE OF THE BEAM'S YOUNG MODULUS

Figure 3 describes the effect of a variation of the Young modulus on some meaningful vibration periods. The three reported diagrams refer to the slowest vibration periods whose vibration modes mainly consist of a bending deformation in the x_1–ζ plane (*case 1*), a bending deformation in x_2–ζ (*case 2*) and a torsion deformation (*case 3*). It should be observed that in the considered system, a perfect uncoupling among axial, bending and twisting vibration modes does not exist. So that the previously defined three cases must be intended in the sense that the elastic energy respectively related to bending in x_1–ζ, bending in x_2–ζ and twisting represents almost the total elastic energy of the system.

The Young modulus varies in the range of $\pm 20\%$ with respect to the reference value E_0 and the ratio $G/E = 0{\cdot}4$ is a constant. Variations of the period T are expressed in percentages with respect to the value $T(E_0)$. For each vibration mode three curves, relative to three different stresses of the cables, are reported. The cable stress τ is expressed by means of the non-dimensional quantity $\bar{\tau} = (\tau/E_0 A) \times 1000$. Here and in what follows it is assumed that $\bar{\tau}$ varies in the range [0·0, 1·0], because this ensures a linear elastic behavior for the material and the fulfilment of the small displacement hypothesis. Figure 3 also contains the numerical values of the periods measured for the reference elastic

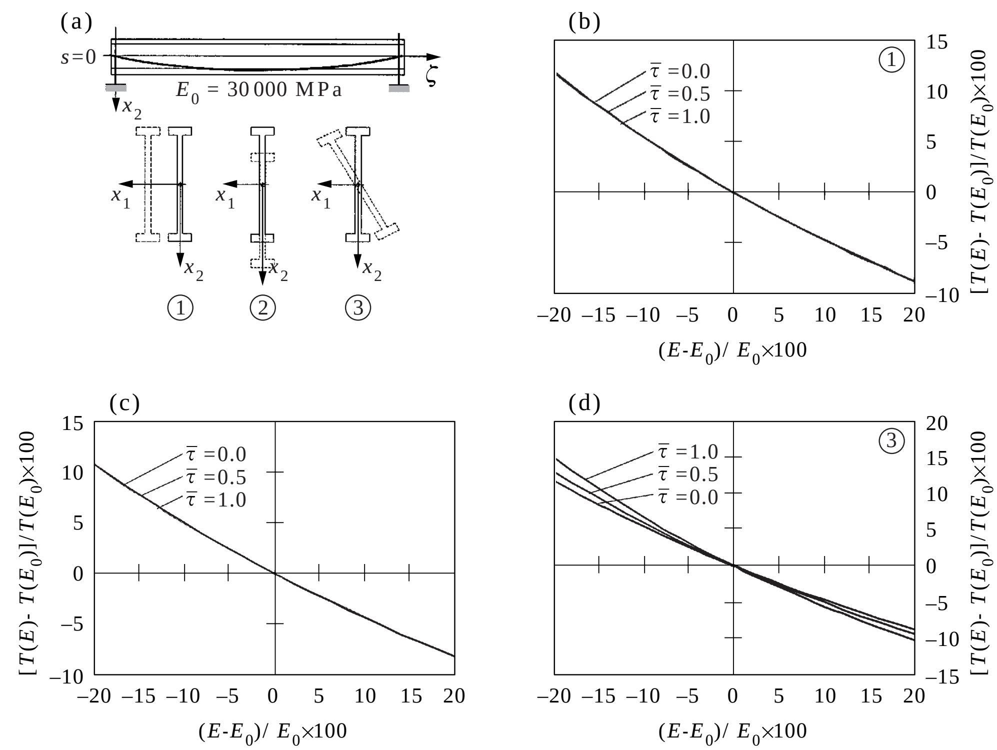

Figure 3. Effects of elastic modulus variations on system free vibrations. $\bar{\tau} = 0{\cdot}0$, $T(E_0) = 0{\cdot}2418$ s (b), 0·0218 s (c), 0·0306 s (d); $\bar{\tau} = 0{\cdot}5$, $T(E_0) = 0{\cdot}2420$ s (b), 0·0218 s (c), 0·0319 s (d); $\bar{\tau} = 1{\cdot}0$, $T(E_0) = 0{\cdot}2420$ s (b), 0·0218 s (c), 0·0335 s (d).

modulus. All the applications in Figure 3 are performed with the cable path geometry reported in Figure 2(a) characterised by $s = 0$ (anchorages at the section centroid).

As foreseeable, the Young's modulus of the beam influences all the vibration modes. Such an influence is almost the same for the three modes and reaches a variation of about 10% for a variation of 20% of the Young's modulus. This is related to the constant ratio between G/E but it is also due to the small influence of the elastic, not varying, term depending on the stiffness of the cable. It is however interesting to observe that the curves obtained with different cable stress are practically superimposed in the modes mainly characterised by bending (Figures 3(b) and (c)). Only the twisting mode (Figure 3(d)) shows a sensitivity to the cable stress. This means that the geometrical term related to the beam compressive stress, which provides a reduction in the system stiffness, is balanced by the geometrical term related to the traction on the cable only for the first bending vibration modes.

5.3. INFLUENCE OF STRESS AND PATH OF THE CABLE

The results obtained by varying both the stress and path of the cable, are reported in Figure 4. As in Figure 3 the first bending and twisting vibration modes are studied. Seven curves are reported for different cable paths obtained by varying s between $-0{\cdot}5$ (parabolic path with anchorage at the top flange of

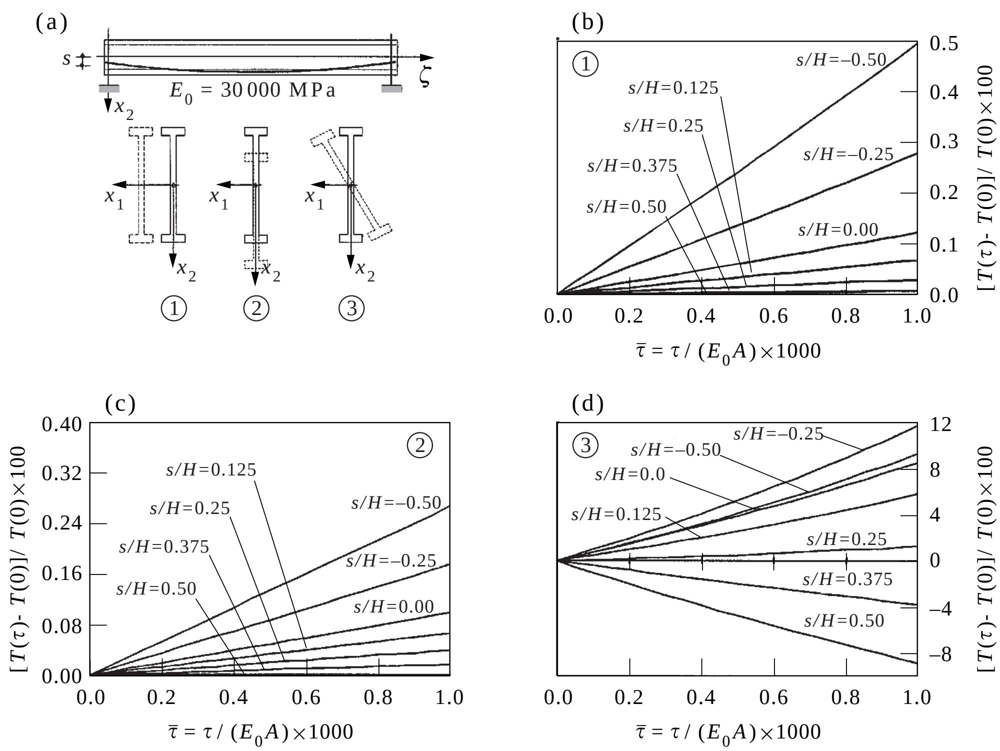

Figure 4. Effects of cable force and cable path variations on system free vibrations. (b) $T(0) = 0{\cdot}2418$ s. (c) $S/H = -0{\cdot}50$, $T(0) = 0{\cdot}0221$ s; $S/H = 0{\cdot}50$, $T(0) = 0{\cdot}0214$ s. (d) $T(0) = 0{\cdot}0306$ s.

the end cross-sections) and 0·5 (straight cable running along the bottom flange of the beam). The cable stress—expressed by the non-dimensional quantity $\bar{\tau} = (\tau/E_0 A) \times 1000$—is assumed to vary in the range 0·0 to 1·0. All the applications are performed adopting the reference value E_0 of the Young's modulus of the beam. Variations of the period are expressed in percentages with respect to the reference value $T(0)$ obtained with a zero cable stress (non-prestressed beam). The numerical value of $T(0)$ is reported in the figure; it is not affected by the cable profile in case 1 (bending vibrations on the plane x_1–ζ) and case 3 (twisting vibrations) because their deformation modes involve displacements which are orthogonal to the plane where the cable lies and then they do not produce a cable elongation. On the contrary, $T(0)$ depends on the cable profile in case 2 (bending vibrations on the plane x_2–ζ) because deformation provides elongations of the cable. This effect is however small, as can be deduced from the extreme values reported in the figure.

In general, it is possible to observe a significant difference between results concerning the bending vibration modes and those relative to the twisting vibration mode. The periods relative to the first two are in fact only slightly modified by variations of cable stress and cable profile and maximum variations of about 0·4% have been observed under the cable stress $\bar{\tau} = 1$ (Figures 4(b) and (c)). Differently, the period relative to the twisting mode reaches variations about 30 times higher (Figure 4(d)). Since the vibration period is related to the stiffness of the vibrating system, under a different point of view it can be concluded that even if the bending stiffness is not appreciably modified by the cable stress and path, the twisting stiffness is strongly influenced.

By considering the effect of the path of the cable, it can be observed that not all the paths produce the same effect and a particular path exists for which the period does not vary even by varying the prestressing force. Such a path, which can be denoted as *indifferent path*, is not the same in the three cases considered and it corresponds to the value $s/H = 0{\cdot}5$ (straight cable) in the case of bending vibrations and approximately to the value $s/H = 0{\cdot}275$ in the case of twisting vibrations.

In the case of bending, the vibration period, always increases by varying the anchorages position (decreasing s) or, in other words, the system stiffness decreases. Such a relation is a monotonic one and reduction of s always produces an increase of the period.

A different and more complex behavior is observed for twisting vibrations. The prestressing produces a reduction of the vibration period when anchorages are disposed with s in the range [0·275, 0·5] while an increase of the period can be observed when s is less than 0·275. Furthermore, the relation between the period and the parameter s is no more monotonic in this case and a reduction of the parameter s produces an increment of period in the range [−0·375, 0·5] while an opposite trend can be observed in the remaining set of anchorage positions [−0·5, −0·375]. This latter aspect is more clearly described by Figure 5, where a three-dimensional representation reports the variations of the twisting period as a continuous function of two variables: the cable prestressing force $\bar{\tau}$ and the anchorage position s/H. Figure 5(a) refers to the first twisting eigenperiod while

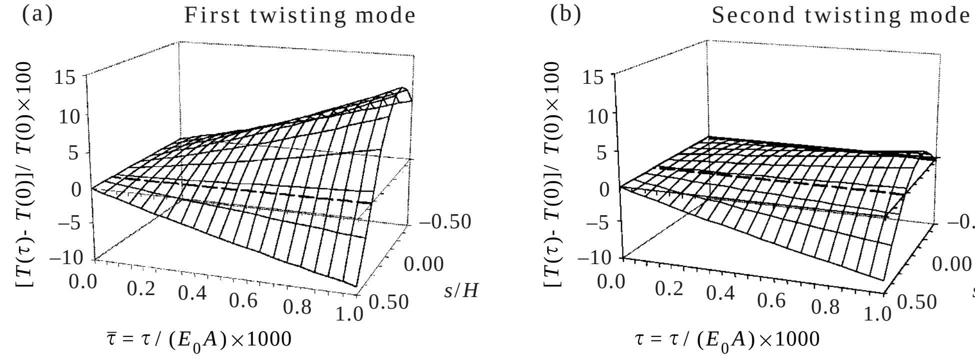

Figure 5. Three-dimensional representation of the period variation for the first two twisting modes.

Figure 5(b) refers to the second twisting eigenperiod; the dashed line denotes the position of the indifferent path. A similar trend has been observed in both cases even if the two vibration modes have different numerical values and the indifferent path changes.

The described characteristic aspects of the problem, as the possible existence of indifferent paths and the possibility of reducing or increasing the system stiffness by varying the cable path, have been observed even in faster vibration modes.

5.4. SIMPLIFED KINEMATICAL MODEL

In Table 2, the first three periods of bending vibrations on the x_2–ζ plane provided by the *KV model* and by the more refined *LS model* are reported. Three beams characterised by different ratios L/H are considered and both cases of prestressing with straight and curved cables with non-dimensional tension $\bar{\tau} = (\tau/E_0 A) \times 1000 = 1$ are investigated. The kinematical *KV model* is based on the following reduced displacement field [14, 15]:

$$\mathbf{u}(\mathbf{x}, \zeta;\ t) = \mathbf{u}_0(\zeta;\ t) - (\mathbf{A}_3 \otimes \mathbf{x})\mathbf{u}'_0(\zeta;\ t) + \boldsymbol{\varphi}(\zeta;\ t)\mathbf{A}_3 \times (\mathbf{x} - \mathbf{x}_s) + \varphi'(\zeta;\ t)\psi(\mathbf{x})\mathbf{A}_3$$
$$(\mathbf{x}, \zeta;\ t) \in V \times [t_0, \infty). \quad (31)$$

As expected, due to the greater model stiffness, vibration periods obtained with the *KV model* are lower than those given by the *LS model*. The difference, almost negligible for the slowest mode of slender beams (less than 1% for $L/H = 15$) becomes very important for beams with a lower ratio L/H (around 6% for $L/H = 5$ and $s/H = 0{\cdot}5$). Furthermore, major differences exist between period values relative to higher vibration modes and differences up to 22% can be observed. In many cases the differences have the same order of the variation of period deriving from variation of the cable force, so that the adoption of the proposed model is crucial in obtaining reliable results in model analysis.

Similar results, not reported here for the sake of brevity, are obtained for twisting and bending vibrations in the x_1–ζ plane because of the geometry of the cross-section adopted. In particular, for I-beams characterised by wither flanges

TABLE 2

Comparison between results obtained by the KV model and the LS model

		Straight cable ($s/H = 0{\cdot}5$)			Curved cable ($s/H = -0{\cdot}5$)		
L/H	Mode	*KV model*	*LS model*	%	*KV model*	*LS model*	%
15	I°	0·02121	0·02140	−0·89	0·02202	0·02217	−0·68
	II°	0·00567	0·00585	−3·08	0·00568	0·00585	−2·91
	III°	0·00254	0·00270	−5·93	0·00255	0·00271	−5·90
10	I°	0·00946	0·00966	−2·07	0·00983	0·00997	−1·40
	II°	0·00256	0·00272	−5·88	0·00256	0·00273	−6·23
	III°	0·00117	0·00132	−11·4	0·00117	0·00132	−11·4
5	I°	0·00242	0·00259	−6·53	0·00251	0·00265	−5·28
	II°	0·00069	0·00082	−15·9	0·00069	0·00082	−15·9
	III°	0·00034	0·00044	−22·7	0·00034	0·00044	−22·7

and T-beams, the adoption of a refined model, as the *LS model*, becomes very important in having good analysis results.

6. CONCLUSIONS

The motion of a thin-walled beam with an internal slipping cable was analyzed in the range of small displacement theory. The balance conditions for motion in the neighbourhood of a configuration in which a stress state due to prestressing and external force is present, were stated in variational form.

The modal analysis was performed for a case with technical interest: a beam prestressed by a parabolic cable.

Developed application provided useful indications on the mechanical characteristics of the system and permitted concluding with the following remarks.

The presence of a stretched cable can make the vibration modes slower or faster and this depends on the cable path geometry.

The cable force and the cable profile selectively affect a reduced set of modes while some of them may be unaffected for every value of the prestressing force. More precisely, for the single vibration mode an *indifferent path* may exist in the family of technically admissible paths. In this case the period of the considered vibration mode is not affected by the cable stress at all, i.e., its sensitivity to the cable stress is null.

In the analyzed case the sensitivity of vibration modes with respect to the cable force usually increases for smaller eigenperiods.

Thin walled beams require the adoption of kinematical models accurately describing warping due to shear and simplified models, such as the Kirchhoff–Vlasov model, providing errors of the same order as the variation of periods induced by variation in the cable force.

The presence of vibration modes almost unaffected by the cable force, or more generally the large differences in the sensitivity of the single vibration modes on the cable traction force, may be of interest in structural identification because this permits separately identifying the constitutive properties of the beam from those modes unaffected by the cable force and identifying in a second step the cable force from the remaining modes.

An alternative application of the presented results may consist in obtaining an active control of the dynamical response of the structure by introducing an adequate cable force.

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